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By Arthur W. Goldstein

SUMMARY

The stability of a horizontal layer of fluid in an accelerated container heated unsteadily from below was investigated theoretically, assuming an incompressible fluid with small density changes resulting from heating. The critical Rayleigh numbers based on the over-all density differential are much higher than for the static case, are dependent only on the density distribution and instantaneous value of the acceleration, and are independent of Prandtl number and rate of change of temperature and body force field. The initial motion corresponds to approximately the same cell shape as for the static case. Rate of transition of temperature perturbations from a stable to an unstable condition is proportional to the rate of increase of the temperature gradient in the region well removed from the walls; rate of transition of the slow motion is proportional to the Prandtl number and rate of increase of the body force field.

INTRODUCTION

In devices that undergo transient heating and require transient cooling, the use of liquid heat sinks has been suggested. If a liquid conductor is used to conduct heat from below to a sink above, or if the sink material is melted by application of heat from below, so that in either case there exists eventually a horizontal layer of fluid, then an unstable situation arises because a colder and denser layer of fluid overlays a heated and less dense layer in the gravitational field. These layers tend to reverse their positions, and the circulatory motion that thus arises provides a substantial increase in the effective heat conduction by the fluid. This motion does not begin immediately, and it is therefore of interest to find at what time it does begin.

The related problem of stability of a horizontal layer with steady and constant temperature gradient (arrived at by very slow heating) has received considerable attention since the original theoretical work of Rayleigh (ref. 1) for free boundaries. Reference 1 showed that, instead of motion being initiated whenever a cold layer overlays a warm layer, a certain critical temperature difference is required to overcome the viscous drag and the heat conduction (which acts to eliminate the motive force derived from thermal gradients). The stability criterion is the value of the Rayleigh number Ra, which is equal to $c_p g h^3 \rho_0^2 \Delta / \mu\kappa$. Jeffries (refs. 2 and 3) obtained theoretical results for rigid, conducting boundaries as well as free boundaries, while Pellew and Southwell (ref. 4) improved the accuracy and among other results showed that only nonoscillatory perturbations need be considered in establishing the condition of neutral stability that separates the regions of stable and unstable initial configurations. Morton (ref. 5) proved a similar theorem where the static-temperature gradient is assumed nonconstant and when the fluid is free at top and bottom (the original condition of Rayleigh). Theoretical predictions of critical Rayleigh number of 1708 were substantiated experimentally by Schmidt and Milverton (ref. 6), Chandra (ref. 7), Schmidt and Saunders (ref. 8), and Malkus (ref. 9).

When the layer is very thin, then initially a columnar motion occurs (ref. 7); this may be expected when the Rayleigh number is less than 1708 (below which cellular motion does not occur), and the temperature difference is large enough for a density variation Δ of about 2.1 percent (ref. 10). A further increase in the temperature difference will cause the Rayleigh number to reach the critical value, at which time the cellular motion will take place. Thus, the critical height h for observing the columnar mode is

$$0.021\,\frac{c_v g h^3 \rho_0^2}{\mu\kappa} < 1708$$

When the layer thickness is greater than this value, then the columnar mode is not observed.

In the case of a time-varying temperature distribution, where the liquid layer is formed from a melting solid heat sink, the liquid layer is initially thin, and for rapid heating rates the temperature differential will give rise to the columnar mode of motion. Because of the small velocity close to the walls with such a mode of flow, no significant increase in heat-transfer rate can be expected, and this mode is disregarded in the present analysis.

The case treated herein includes a time-variant force field. In a ballistic missile that requires cooling, the varying velocity of the missile will impose a time-varying body force field on the fluid in addition to the gravitational body force field. A criterion for marginal stability is stated and analyzed and used to investigate the effect of unsteady heating and body forces on the stability of the fluid layer.

SYMBOLS

$A(x,y)$	function of x,y, which separates the functional dependences in $\rho = A(x,y)\theta(z,t)$ and $w = A(x,y)\,\omega(z,t)$
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
E_{mn}	$\int_0^1 \overline{\rho}_z \sin m\pi z\, W_n\, dz$
G	total force field resulting from gravity and acceleration
g	acceleration due to gravity (in negative z-direction)
h	height of layer in z^*-direction
K_{mn}	$\int_0^1 \sin m\pi z\, W_n\, dz$
k	unit vector in direction of z^* increasing
m,n	positive integers
Pr	Prandtl number, $c_p\mu/\kappa$
p	pressure
Ra	Rayleigh number, $c_p h^3 G\rho_0[\overline{\rho}^*(h,t^*) - \overline{\rho}^*(0,t^*)]/\mu\kappa$
Ra_0	critical Rayleigh number for static case (lowest mode of motion)
T	temperature of fluid
t	time
t_{cr}	critical time (perturbations pass from stable to unstable regime)
V	velocity of liquid
$W(z)$	function of z in approximate separated form for ω, $\omega(z,t)\approx\omega_0(t)W(z)$
W_n	function of z that separates the functional dependences of $\omega = \Sigma\omega_n(t)W_n(z)$, $(n=1, 2, 3, \text{etc.})$
w	vertical component of motion, $V\cdot k$
x,y,z	coordinates
α	coefficient of thermal expansion, $\frac{1}{\rho_0}\frac{d\rho}{dT} = -\alpha$
β	index of rate of fluid expansion at lower wall (see eq. (39))
γ	ratio of specific heats
Δ	proportional critical density differential, $[\overline{\rho}^*(h,t^*) - \rho_0]/\rho_0$
θ	function of z,t that separates variables in $\rho = A(x,y)\,\theta(z,t)$
θ_n	function of t that separates variables in $\theta(z,t) = \Sigma\theta_n(t)\sin n\pi z$
θ_0	function of t in approximate separated form for θ, $\theta(z,t)\approx\theta_0(t)\tau(z)$
κ	conductivity
λ	parameter indicating shape of circulation cells
μ	viscosity of fluid
ρ	density of fluid
ρ_0	reference density, $\overline{\rho}^*(0,t^*_{cr})$
$\tau(z)$	function of z in approximate separated form for θ, $\theta(z,t)\approx\theta_0(t)\tau(z)$
ω	function of z,t that separates variables in $w = A(x,y)\,\omega(z,t)$
ω_n	function of t that separates variables in $\omega = \Sigma\omega_n(t)\,W_n(z)$
$\omega_0(t)$	function of t in approximate separated form for ω, $\omega(z,t)\approx\omega_0(t)\,W(z)$
—	bar indicates basic configuration, applied to p, T, V, w, and ρ; absence of bar indicates perturbation variables
*	asterisk indicates variables with dimensions, applied to p, T, t, V, w, x, y, z, and ρ; absence of asterisk indicates dimensionless, normalized variables

Examples:

ρ normalized, dimensionless density perturbation, $\rho^*/\rho_0\Delta$

ρ^* perturbation of fluid density

$\bar{\rho}$ normalized dimensionless density in basic configuration $\left(\frac{\bar{\rho}^*}{\rho_0}-1\right)/\Delta$

$\bar{\rho}^*$ density of fluid in basic configuration

Subscripts:

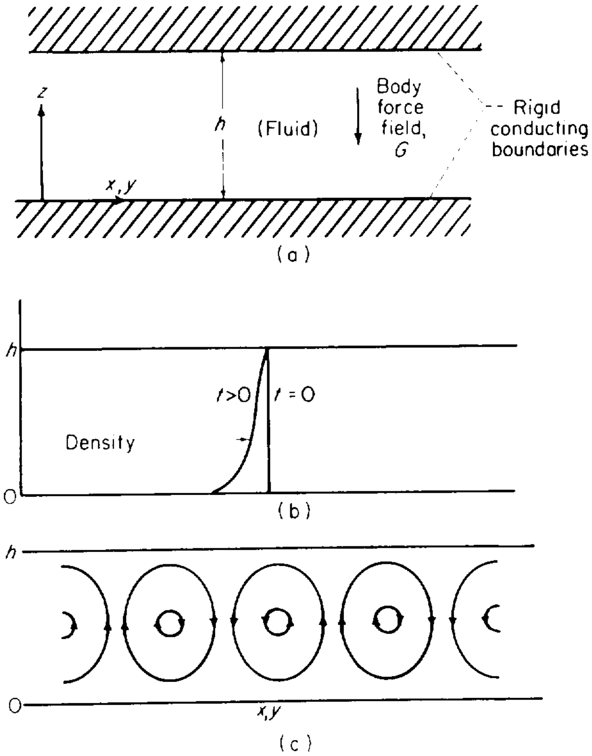
cr value at critical time t_{cr}

z,t differentiation with respect to z and t, respectively

PHYSICAL ASSUMPTIONS

A viscous, conducting, incompressible fluid with constant viscosity, conductivity, and specific heats is assumed to be between two horizontal plates of very large horizontal dimensions compared with the vertical space between. The density, which is supposed independent of the imposed pressure, will vary a small amount throughout the fluid because of temperature variations that may be large for fluids with small thermal coefficients of volume expansion. A steady gravitational field and a vertical acceleration are assumed to be acting on the fluid (fig. 1). Both plates are assumed infinitely conducting. Initially the temperature is assumed uniform, but at some time the bottom plate is heated while the upper plate is kept at a constant temperature at all times. Since the heating is uniform in the horizontal direction, the fluid is expanded vertically. The temperature and density distribution that results from heating the bottom plate and from the fluid conductivity, together with the velocity of vertical expansion, is hereinafter designated as the basic configuration. This consists in a small vertical motion and a temperature distribution for unsteady heating of a conducting slab with a negligible modification resulting from the convection.

Superimposed on this basic configuration is a slow motion in the form of cellular patterns. The velocity is assumed so small that quadratic terms in the equations of motion are considered negligible compared with linear terms. In addition, there is a small perturbation in the basic transient temperature distribution. In the initial stages of the heating, the velocity and temperature perturba-



(a) Configuration, coordinates, and force field.
(b) Basic density distribution.
(c) Pattern of slow cellular motion.

FIGURE 1. Configuration, coordinates, density, and slow motion.

tions will be damped out because of the viscosity and conductivity of the fluid; but at some later time the transient temperature gradient is large enough to drive the fluid in slow motion, and the motion will grow with time until the linear approximation is no longer valid. Briefly, the method of analysis consists in assuming a slow motion and a small perturbation in the initial temperature configuration and finding out whether the perturbation and motion will be damped out in time or will increase indefinitely with time.

EQUATIONS OF MOTION

BASIC-CONFIGURATION AND PERTURBATION EQUATIONS

The equations of motion (momentum, continuity, energy) are expressed with each dependent variable partitioned into a sum of the value for the basic configuration (indicated with a bar $^-$) and the value for the perturbation flow (no bar). Thus, the velocity vector V is written $\bar{V}^*+V^*$, where the asterisk indicates a variable with dimensions. The bar is applied to p, T, V, w, and ρ. The asterisk is applied to p, T, t, V, w, x, y, z, and ρ. Each of the equations is then split into two components by the usual method of perturbations, as follows. First, the equations are assumed to be satisfied by the basic configuration alone (zero perturbation). Secondly, the equation with basic quantities is subtracted from the complete original equation. This second equation is then linearized by discarding terms that are quadratic in slow velocities and perturbation components of the variables, since these are assumed small compared with the linear terms.

In the present case, before establishing the perturbation equations, the coordinate system is transformed from a stationary set to a moving set that is considered to be moving at a variable speed parallel to the vertical or z-axis, as in a space vehicle vertically reentering the atmosphere. The effect of this transformation is to add a field force term to the momentum equations and to leave the continuity and energy equations unmodified. The process described results in the following equations for the basic flow:

Momentum:

$$\bar{\rho}^*\frac{\partial\bar{w}^*}{\partial t^*}+\bar{\rho}^*\bar{w}^*\frac{\partial\bar{w}^*}{\partial z^*}=-\frac{\partial\bar{p}^*}{\partial z^*}-\bar{\rho}^*G+\frac{4}{3}\mu\frac{\partial^2\bar{w}^*}{\partial z^{*2}} \tag{1}$$

Continuity:

$$\frac{\partial\bar{\rho}^*}{\partial t^*}+\bar{\rho}^*\frac{\partial\bar{w}^*}{\partial z^*}+\bar{w}^*\frac{\partial\bar{\rho}^*}{\partial z^*}=0 \tag{2}$$

Energy:

$$\bar{\rho}^*c_v\left(\frac{\partial\bar{T}^*}{\partial t^*}+\bar{w}^*\frac{\partial\bar{T}^*}{\partial z^*}\right)+\bar{\rho}^*\bar{p}^*\left[\frac{\partial}{\partial t^*}\left(\frac{1}{\bar{\rho}^*}\right)+\bar{w}^*\frac{\partial}{\partial z^*}\left(\frac{1}{\bar{\rho}^*}\right)\right]=\kappa\frac{\partial^2\bar{T}^*}{\partial z^{*2}}+\frac{4}{3}\mu\left(\frac{\partial\bar{w}^*}{\partial z^*}\right)^2 \tag{3}$$

In these equations the horizontal (x,y) velocity components have been assumed to be zero, and the x,y derivatives zero by virtue of the uniform conditions in horizontal direction.

The corresponding equations for the perturbation flow are as follows:

Momentum:

$$\bar{\rho}^*\frac{\partial V^*}{\partial t^*}+k\rho^*\frac{\partial\bar{w}^*}{\partial t^*}+\bar{\rho}^*\bar{w}^*\frac{\partial V^*}{\partial z^*}+k\bar{\rho}^*w^*\frac{\partial\bar{w}^*}{\partial z^*}+k\rho^*\bar{w}^*\frac{\partial\bar{w}^*}{\partial z^*}=-\nabla p^*-\rho^*Gk+\mu\left(\nabla^2V^*+\frac{1}{3}\nabla\nabla\cdot V^*\right) \tag{4}$$

Continuity:

$$\frac{\partial\rho^*}{\partial t^*}+\bar{\rho}^*\nabla\cdot V^*+\rho^*\frac{\partial\bar{w}^*}{\partial z^*}+w^*\frac{\partial\bar{\rho}^*}{\partial z^*}+\bar{w}^*\frac{\partial\rho^*}{\partial z^*}=0 \tag{5}$$

Energy:

$$\rho^*c_v\left(\frac{\partial\bar{T}^*}{\partial t^*}+\bar{w}^*\frac{\partial\bar{T}^*}{\partial z^*}\right)+\bar{\rho}^*c_v\left(\frac{\partial T^*}{\partial t^*}+\bar{w}^*\frac{\partial T^*}{\partial z^*}+w^*\frac{\partial\bar{T}^*}{\partial z^*}\right)-\frac{\bar{p}^*}{\bar{\rho}^*}\left(\frac{\partial\rho^*}{\partial t^*}+\bar{w}^*\frac{\partial\rho^*}{\partial z^*}+w^*\frac{\partial\bar{\rho}^*}{\partial z^*}\right)-\left(\frac{p^*}{\bar{\rho}^*}-\frac{\bar{p}^*}{\bar{\rho}^{*2}}\rho^*\right)\left(\frac{\partial\bar{\rho}^*}{\partial t^*}+\bar{w}^*\frac{\partial\bar{\rho}^*}{\partial z^*}\right)=\kappa\nabla^2T^*+4\mu\frac{\partial\bar{w}^*}{\partial z^*}\left(\frac{dw^*}{\partial z^*}-\frac{1}{3}\nabla\cdot V^*\right) \tag{6}$$

where k is the unit vector parallel to the z-axis.

The boundary conditions for the basic flow are

$$\bar{w}^*=0,\ \bar{T}^*=\text{function of } t^*,\ \text{at } z^*=0$$

$$\bar{p}^*=p_1,\ \bar{w}^*=\mathrm{d}h/\mathrm{d}t^*,\ \bar{T}^*=T_1,\ \text{at } z^*=h$$

and initial conditions $\bar{T}^*=T_1=\text{constant}$. For the perturbations, V^* is zero at either wall by the no-slip condition, and because of the assumed large wall conductivity, $T^*=0$ also. That is,

$$V^*=0,\ T^*=0,\ \text{at } z^*=0$$

$$V^*=0,\ T^*=0,\ \text{at } z^*=h$$

Because of the over-all expansion of the fluid, the upper boundary will be displaced and h will vary with time. The variation is neglected because of the assumption that the fluid expansion is small.

Temperature variations may be expressed in terms of density variations by means of the thermal coefficient of density change α:

$$\alpha=-\frac{1}{\rho_0}\frac{\mathrm{d}\bar{\rho}^*}{\mathrm{d}\bar{T}^*}=-\frac{1}{\rho_0}\frac{\mathrm{d}\rho^*}{\mathrm{d}T^*} \tag{7}$$

where ρ_0 is some standard density of the fluid. Thus, the basic-flow energy equation (3) is reducible to

$$\frac{\overline{\rho}^* c_v}{\alpha\rho_0}\left[1+\frac{\overline{p}^*\alpha\rho_0}{c_v(\overline{\rho}^*)^2}\right]\left(\frac{\partial\overline{\rho}^*}{\partial t^*}+\overline{w}^*\frac{\partial\overline{\rho}^*}{\partial z^*}\right)=\frac{\kappa}{\alpha\rho_0}\frac{\partial^2\overline{\rho}^*}{\partial z^{*2}}-\frac{4}{3}\mu\left(\frac{\partial\overline{w}^*}{\partial z^*}\right)^2$$

while that for the perturbation flow (eq. (6)) becomes

$$\frac{\overline{\rho}^* c_v}{\alpha\rho_0}\left[1+\frac{\overline{p}^*\alpha\rho_0}{c_v(\overline{\rho}^*)^2}\right]\left(\frac{\partial\rho^*}{\partial t^*}+\overline{w}^*\frac{\partial\rho^*}{\partial z^*}+w^*\frac{\partial\overline{\rho}^*}{\partial z^*}\right)+\frac{\rho^* c_v}{\alpha\rho_0}\left[1-\frac{\alpha\rho_0\overline{p}^*}{c_v(\overline{\rho}^*)^2}+\frac{\alpha\rho_0 p^*}{c_v\overline{\rho}^*\rho^*}\right]\left(\frac{\partial\overline{\rho}^*}{\partial t^*}+\overline{w}^*\frac{\partial\overline{\rho}^*}{\partial z^*}\right)=\frac{\kappa}{\alpha\rho_0}\nabla^2\overline{\rho}^*-4\mu\frac{\partial\overline{w}^*}{\partial z^*}\left(\frac{\partial w^*}{\partial z^*}-\frac{1}{3}\nabla\cdot V^*\right)$$

In both energy equations there occurs the group $\alpha\rho_0\overline{p}^*/c_v(\overline{\rho}^*)^2$, which (since the density changes are small) is approximately $\alpha\overline{p}^*/c_v\overline{\rho}^*$. Since $\overline{p}^*$ is of the order of $\rho_0 Gh$, the group is approximately $\alpha Gh/c_v$. For water in a normal gravitational field, $\alpha g/c_v\approx 0.5\times 10^{-8}$/cm. For mercury, $\alpha g/c_v\approx 1.3\times 10^{-7}$/cm. Therefore, there is a large class of problems for which $\alpha Gh/c_v$ may be considered negligible, and for this class the energy equations are modified by discarding this term in comparison with 1.0. By similar reasoning, the term $\alpha\rho_0 p^*/c_v\overline{\rho}^*\rho^*$ may also be discarded, provided p^* is assumed to be of the order of $\rho^* Gh$. Ostrach (ref. 11) pointed out that this term represents the ratio of compression work to heating energy and is bound to be small for fluids of small assumed thermal coefficient of volume expansion but may be large for fluids near the critical state where α is large and in large gravitational or accelerational fields such as in rotating machinery where fields of 10^5 g are possible. The two energy equations now assume the following forms:

$$\frac{\partial\overline{\rho}^*}{\partial t^*}+\overline{w}^*\frac{\partial\overline{\rho}^*}{\partial z^*}=\frac{\kappa}{\overline{\rho}^* c_v}\frac{\partial^2\overline{\rho}^*}{\partial z^{*2}}-\frac{4}{3}\frac{\mu\alpha\rho_0}{\overline{\rho}^* c_v}\left(\frac{\partial\overline{w}^*}{\partial z^*}\right)^2 \tag{8}$$

and

$$\frac{\partial\rho^*}{\partial t^*}+\overline{w}^*\frac{\partial\rho^*}{\partial z^*}+w^*\frac{\partial\overline{\rho}^*}{\partial z^*}+\frac{\rho^*}{\overline{\rho}^*}\left(\frac{\partial\overline{\rho}^*}{\partial t^*}+\overline{w}^*\frac{\partial\overline{\rho}^*}{\partial z^*}\right)=\frac{\kappa}{\overline{\rho}^* c_v}\nabla^2\rho^*-\frac{4\mu\alpha\rho_0}{\overline{\rho}^* c_v}\frac{\partial\overline{w}^*}{\partial z^*}\left(\frac{\partial w^*}{\partial z^*}-\frac{1}{3}\nabla\cdot V^*\right) \tag{9}$$

ORDER OF MAGNITUDE OF TERMS AND SCALING FACTORS

Basic configuration.—In order to simplify the equations of motion, it is necessary to investigate the order of magnitude of the various terms in the equations so that the negligible terms may be eliminated. For this purpose the variables must be scaled to new dimensionless variables of order 1, so that the various terms may be compared. The obvious choice of scale for the lengths is h:

$$\left.\begin{aligned}x^*&=xh\\ y^*&=yh\\ z^*&=zh\end{aligned}\right\} \tag{10}$$

The density variation is small; therefore, if ρ_0 is the density attained at the lower wall at the time t^*_{cr} when the instability develops, the dimensionless density $\overline{\rho}(z,t)$ for the basic motion is given by

$$\overline{\rho}^*=\rho_0(1+\overline{\rho}\Delta) \tag{11}$$

$$\rho_0\equiv\overline{\rho}^*(0,t^*_{cr}) \tag{11a}$$

where

$$\overline{\rho}(h,t^*)=1,\ \overline{\rho}(0,t^*_{cr})=0,\ \Delta\equiv\frac{\overline{\rho}^*(h,t^*)}{\rho_0}-1 \tag{11b}$$

and $\Delta\ll 1$. If relations (10), (11), $\overline{w}^*=\overline{w}a$, and $t^*=tb$ are set into the continuity equation (2) (a,b are as yet undetermined constants), and it is assumed that $\overline{w}$ and t are dimensionless and of order 1, then there results

$$\frac{\Delta h}{ab(1+\Delta\overline{\rho})}\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\overline{w}}{\partial z}+\frac{\Delta}{1+\Delta\overline{\rho}}\overline{w}\frac{\partial\overline{\rho}}{\partial z}=0$$

Since Δ is assumed small, the last term is negligible compared with the second, and the equation reduces to

$$\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\overline{w}}{\partial z}=0 \tag{12}$$

when $ab=\Delta h$ is assumed.

The energy equation (8) is normalized by substituting the dimensionless variables of order 1.0:

$$\frac{4\alpha\mu\Delta}{3\rho_0 c_v b}\left(\frac{\partial\overline{w}}{\partial z}\right)^2-\frac{\kappa b}{\rho_0 c_v h^2}\frac{\partial^2\overline{\rho}}{\partial z^2}=-\frac{\partial\overline{\rho}}{\partial t}-\overline{w}\frac{\partial\overline{\rho}}{\partial z}\Delta$$

The conduction term $\nabla^2\overline{\rho}$ and the heat rate term $\partial\overline{\rho}/\partial t$ are comparable if it is assumed that $b=h^2\rho_0 c_v/\kappa$. This is equivalent to choosing a time scale comparable with that for conduction of temperature disturbances from one wall to the other. The velocity and time scales then result in the dimensionless variables $\overline{w}$ and $\overline{t}$, given by

$$\left.\begin{aligned} \overline{w}^* &= \frac{\kappa\Delta}{\rho_0 c_v h}\overline{w} \\ t^* &= \frac{\rho_0 c_v h^2}{\kappa} t \end{aligned}\right\} \tag{13}$$

With these substitutions the first term of the energy equation gives the coefficient $\frac{4\alpha\mu\kappa\Delta}{3h^2\rho_0^2c_v^2}$, which is equal to $\frac{4}{3}\left(\frac{\alpha G_{cr}h}{c_p}\right)\Delta^2\Big/\left(Ra_{cr}\frac{c_v}{c_p}\right)$, where G_{cr} is the value of G at $t=t_{cr}$, the time when instability is attained, and Ra_{cr} is the Rayleigh number at $t=t_{cr}$, $c_pG_{cr}h^3\rho_0^2\Delta/\kappa\mu$. Since Ra_{cr} is of order 2000 or larger in the range of interest, $\Delta\ll 1$, and $\alpha G_{cr}h/c_p\ll 1$, the first (dissipation) term in the energy equation may be discarded. Also, the last term may be discarded because $\Delta\ll 1$, so that the energy equation for the basic flow is

$$\nabla^2\overline{\rho}=\frac{\partial\overline{\rho}}{\partial t} \tag{14}$$

which is the equation for transient conductive heating of a slab. The boundary and initial conditions are $\overline{\rho}(0,t)=f(t)$, $\overline{\rho}(1,t)=1$, $\overline{\rho}(z,0)=1$, and $f(t_{cr})=0$.

With the new variables, the momentum equation for the basic flow is

$$\frac{\partial\overline{p}}{\partial z}=-(1+\overline{\rho}\Delta)\frac{G}{G_{cr}}+\frac{(c_p/c_v)\Delta^2}{Ra_{cr}}\left[\frac{4}{3}\frac{\partial^2\overline{w}}{\partial z^2}-\frac{c_p/c_v}{Pr}(1+\overline{\rho}\Delta)\left(\overline{w}\Delta\frac{\partial\overline{w}}{\partial z}+\frac{\partial\overline{w}}{\partial t}\right)\right]$$

where

$$\overline{p}=\overline{p}^*/\rho_0G_{cr}h$$

Within the limits of the approximations, the solution is

$$\overline{p}=\overline{p}_1+\frac{(1-z)G}{G_{cr}}$$

This pressure distribution results from the body force field alone.

Order of magnitude and scaling factors in perturbation equations.—Scaling of the slow-motion and temperature perturbation equations differs somewhat from the corresponding process in the basic-configuration equations. In the latter, the velocity resulted from the expansion of the fluid, and consequently the continuity equation is the proper one for seeking a new variable $\overline{w}$ of order unity. In the perturbation equations, the velocity results from the gravitational field acting on the fluid and is impeded by the viscous effect. Thus, V should be scaled from these two terms in the equation of motion. The velocity of order unity is then

$$\left.\begin{aligned} V &\equiv \frac{\mu}{h^2G_{cr}\rho_0\Delta}V^* \\ w &\equiv \frac{\mu}{h^2G_{cr}\rho_0\Delta}w^* \end{aligned}\right\} \tag{15}$$

With

$$\left.\begin{aligned} p &\equiv \frac{p^*}{hG_{cr}\rho_0\Delta} \\ \rho &\equiv \frac{\rho^*}{\rho_0\Delta} \end{aligned}\right\} \tag{16}$$

the equation of motion becomes

$$\begin{aligned}\frac{\gamma}{Pr}\Big[(1-\Delta\overline{\rho})\frac{\partial V}{\partial t}&+k\frac{\Delta^2\gamma}{Ra_{cr}}\rho\frac{\partial\overline{w}}{\partial t}+\Delta(1+\overline{\rho}\Delta)\overline{w}\frac{\partial V}{\partial z} \\ &+k\Delta(1+\overline{\rho}\Delta)w\frac{\partial\overline{w}}{\partial z}+k\frac{\gamma\Delta^3}{Ra_{cr}}\rho\overline{w}\frac{\partial\overline{w}}{\partial z}\Big] \\ &=-\nabla p-k\rho\frac{G}{G_{cr}}+\nabla^2V+\frac{1}{3}\nabla\nabla\cdot V\end{aligned} \tag{17}$$

With the same substitutions, the continuity equation is

$$\nabla\cdot V=-\frac{\Delta}{1+\overline{\rho}\Delta}\left\{w\frac{\partial\overline{\rho}}{\partial z}+\frac{\gamma}{Ra_{cr}}\left[\frac{\partial\rho}{\partial t}+\Delta\left(\rho\frac{\partial\overline{w}}{\partial z}+\overline{w}\frac{\partial\rho}{\partial z}\right)\right]\right\} \tag{18}$$

and the energy equation,

$$\begin{aligned}-2\Delta\frac{\alpha hG_{cr}}{c_v}\left(\frac{\partial w}{\partial z}-\frac{1}{3}\nabla\cdot V\right)\frac{\partial w}{\partial z}+\nabla^2\rho &=(1+\overline{\rho}\Delta)\left(\frac{\partial\rho}{\partial t}+\Delta\overline{w}+\frac{\partial\rho}{\partial z}+\frac{Ra_{cr}w}{\gamma}\frac{\partial\overline{\rho}}{\partial z}\right) \\ &+\rho\Delta\left(\frac{\partial\overline{\rho}}{\partial t}+\Delta\overline{w}\frac{\partial\overline{\rho}}{\partial z}\right)\end{aligned} \tag{19}$$

With the approximations $\Delta\ll 1$ and $\gamma\approx 1$, the perturbation flow satisfies the following equations:

Motion:

$$\frac{1}{Pr}\frac{\partial V}{\partial t}=-\nabla p-k\rho\frac{G}{G_{cr}}+\nabla^2 V+\frac{1}{3}\nabla\nabla\cdot V \qquad (20)$$

Continuity:

$$\nabla\cdot V=0 \qquad (21)$$

Energy:

$$\nabla^2\rho=\frac{\partial\rho}{\partial t}+Ra_{cr}w\frac{\partial\bar{\rho}}{\partial z} \qquad (22)$$

These perturbation equations involve only $\bar{\rho}_z$ from the basic configuration, and this may be obtained from equation (14) alone, which does not involve $\bar{w}$. It is not necessary to find the velocity distribution of the basic flow, and the equations of continuity and momentum need not be used.

STABILITY CONDITION FROM PERTURBATION EQUATIONS

Following Rayleigh (ref. 1), the pressure and velocity components $V\cdot i$ and $V\cdot j$ are eliminated from the perturbation equation of motion (eq. (20)) by first taking the divergence and then the derivative with respect to z. Secondly, the Laplacian of the vertical component is subtracted, and the continuity relation (eq. (21)) is employed, yielding

$$\frac{1}{Pr}\nabla^2\frac{\partial w}{\partial t}=\left(\frac{\partial^2\rho}{\partial z^2}-\nabla^2\rho\right)\frac{G}{G_{cr}}+\nabla^2\nabla^2 w \qquad (23)$$

The x and y variables are separated out by the following assumptions (ref. 4):

$$\left.\begin{aligned}&\rho=A(x,y)\theta(z,t)\\&w=A(x,y)\omega(z,t)\\&\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\lambda^2 A=0\end{aligned}\right\} \qquad (24)$$

The function $A(x,y)$ is periodic, and gives rise to circulation cells if $\lambda^2>0$. The parameter λ indicates the shape of the cells, as can be easily seen for the case of two-dimensional cells where $\partial^2 A/\partial y^2=0$. Then $A=\sin\lambda x$ or $A=\cos\lambda x$. The complete cell width is then $x^*/h=x=2\pi/\lambda$. Thus, λ indicates the ratio of cell height to cell width.

With these assumptions, equation (23) reduces to

$$\left(\frac{\partial^2}{\partial z^2}-\lambda^2\right)\left(\frac{\partial^2}{\partial z^2}-\lambda^2-\frac{1}{Pr}\frac{\partial}{\partial t}\right)\omega=-\lambda^2\theta\frac{G}{G_{cr}} \qquad (25)$$

and the energy equation (22) reduces to

$$\left(\frac{\partial^2}{\partial z^2}-\lambda^2-\frac{\partial}{\partial t}\right)\theta=Ra_{cr}\omega\bar{\rho}_z \qquad (26)$$

The previously given boundary conditions for the temperature perturbations reduce to

$$\theta(0,t)=\theta(1,t)=0 \qquad (27)$$

The continuity equation (21) permits the boundary conditions on the velocity perturbation vector V to be restated in terms of the derivative $\partial w/\partial z$ at the boundary. The complete set of boundary conditions on the velocity perturbation then assumes the form

$$\omega(0,t)=\omega(1,t)=\frac{\partial\omega(0,t)}{\partial z}=\frac{\partial\omega(1,t)}{\partial z}=0 \qquad (28)$$

A CRITERION FOR MARGINAL STABILITY

Because of the homogeneous form of the differential equations and the boundary conditions at $z=0,1$, the solution of these equations is any of a number of eigenfunctions, each associated with a corresponding value of the Rayleigh number Ra_{cr}. Furthermore, homogeneous conditions are required at certain values of t to establish solutions of this type. Because of the order of differentiation with respect to t, it is possible to select two such conditions.

Some physical insight will help establish these conditions. As the temperature of the bottom wall is increased, one may imagine various perturbations being imposed on the fluid. Initially, when the temperature difference is small, the perturbations will be damped out. There will come a time when one mode of motion will continue with stationary amplitude and then grow at later times. This time of critical or marginal stability is designated as the critical time t_{cr}. It is reasonable to determine this time in terms of conditions related only to time in the immediate vicinity of $t=t_{cr}$, since any growing motion may be damped out later by appropriately scheduling the temperature of the lower wall. Furthermore, it is also appropriate to require that both the slow-motion velocity and the temperature distribution shall have growing amplitudes before the temperature distribution can be regarded as unstable, as the following example indicates.

Assume a static linear temperature distribution known to be stable. If a small velocity is imposed on the fluid with zero temperature perturbation, then the temperature perturbation will initially grow from zero before decaying. Similarly, it is possible to have a period with a growing velocity perturbation during a condition of fluid stability. Thus, instability requires that the growth rates for both the velocity amplitude and temperature amplitude be equal to or greater than zero. If this requirement is combined with that for the definition of the critical time, then an eigenfunction condition is also satisfied by setting

$$\frac{\partial\omega}{\partial t}=0, (t=t_{cr})$$

It is also required that $\frac{1}{\theta}\frac{\partial\theta}{\partial t}\geqq 0$ at $t=t_{cr}$, and a second homogeneous condition be satisfied. Both requirements are satisfied by setting

$$\frac{\partial\theta}{\partial t}=0, (t=t_{cr})$$

This second condition also serves the purpose of defining the least stable perturbation, since the fluid instability and the value $\frac{1}{\theta}\frac{\partial\theta}{\partial t}$ may be expected to increase with the increase in temperature of the lower wall. Thus, the condition $\frac{\partial\theta}{\partial t}=0$ $(t=t_{cr})$ will result in a minimum critical Rayleigh number.

RELATION TO QUASI-STEADY SOLUTION, OSCILLATIONS

When equations (25) and (26) are solved at the critical time $(t=t_{cr})$, use of the preceding conditions for marginal stability yields the simplified forms

$$\left(\frac{\partial^2}{\partial z^2}-\lambda^2\right)^2\omega=-\lambda^2\theta \qquad (t=t_{cr}) \tag{29}$$

$$\left(\frac{\partial^2}{\partial z^2}-\lambda^2\right)\theta=Ra_{cr}\omega\bar{\rho}_z \qquad (t=t_{cr}) \tag{30}$$

Since $t=t_{cr}$ may be regarded as a constant parameter to be determined from these equations, the equations differ from those for the steady-state case essentially in that $\bar{\rho}_z$ is a function of z. In the steady case, the solution for $\bar{\rho}$ results in $\bar{\rho}_z=1$. If the heating is slow, the curvature of the $\bar{\rho}$ profile is small, and $\bar{\rho}_z=1$ will represent a good approximation in the mean. In this case,

$$Ra_{cr}\approx Ra_0=1707.8$$

where Ra_0 is the critical Rayleigh number for the steady case.

In the present case, $\bar{\rho}_z$ is not constant, and equations (29) and (30) are equivalent to the investigation of the stability of the density profile $\bar{\rho}$ as it stands at any instant, neglecting variations of $\bar{\rho}$ with time. This situation exists if the time scale for the temperature change in the system is large compared with the time scale for instability to develop. This point was made by Morton (ref. 5), who investigated the steady-state case with a nonconstant temperature gradient and shear-free upper and lower surfaces. To use this approach from the beginning, the time variations of G and $\bar{\rho}_z$ would be neglected in equations (25) and (26). Then the z and t variables may be separated by use of exponentials into a set of functions

$$\omega_n=e^{\alpha_n t}W_n(z)$$

$$\theta_n=e^{\alpha_n t}\varphi_n(z)$$

For any particular eigenfunction pair $(\varphi_n(z), W_n(z))$, which is at the state of critical stability $(\alpha_n=0)$ the equations of motion (25) and (26) then reduce to (29) and (30). The eigenvalues for Ra_{cr} that result are therefore the same for both methods.

The development of instability may be expected to involve only nonoscillatory motions from the following considerations. If the quasi-steady approximation is assumed sufficient for the description of the motion near the critical time, then the argument of Pellew and Southwell (ref. 4) indicates that, for a straight static-density curve, the Rayleigh number is negative for oscillatory motions, and the rate of growth is also negative. That is, the fluid configuration is stable if an oscillatory slow motion is to exist. For nonlinear static-density distributions, some deviation of $\bar{\rho}_z$ from 1.0 can be imposed on the density distribution before the rate of growth changes from a negative to a positive value, if such a change takes place at all. Thus, the continuous nature of the variation of the Rayleigh number and of the perturbation growth rate with variations in density distribution indicates that oscillatory motions will exist only under stable conditions for a nontrivial class of density distributions.

VARIATION OF STABILITY NEAR CRITICAL TIME

At the critical time $(\partial\omega/\partial t=\partial\theta/\partial t=0)$, the question arises whether the perturbations reach a maximum value (transition from unstable to stable equilibrium), or reach a minimum. From physical considerations, transition is from the stable to the unstable regime if the Rayleigh number based on instantaneous values of the over-all density difference and the instantaneous field force G is increasing. The change of the profile shape from a more stable shape to a more unstable shape will have a similar effect.

An approximate set of analytic conditions for the assurance that the fluid is passing into an unstable state may be obtained if it is assumed that the solution to the differential equations (25) and (26) may be approximated by the function pair

$$\omega=\omega_0(t)W(z)$$

$$\theta=\theta_0(t)\tau(z)$$

in the region $t\approx t_{cr}$. The differential equations (25) and (26) are, with these substitutions,

$$\theta_0(\tau_{zz}-\lambda^2\tau)-\frac{d\theta_0}{dt}\tau=Ra_{cr}\omega_0\bar{\rho}_z W$$

$$\omega_0\left(\frac{d^2}{dz^2}-\lambda^2\right)^2W-\frac{1}{Pr}\frac{d\omega_0}{dt}\left(\frac{d^2}{dz^2}-\lambda^2\right)W=-\lambda^2\frac{G}{G_{cr}}\theta_0\tau$$

When these are differentiated with respect to time,

$$\frac{d\theta_0}{dt}(\tau_{zz}-\lambda^2\tau)-\frac{d^2\theta_0}{dt^2}\tau=Ra_{cr}W\left(\omega_0\bar{\rho}_{zt}+\bar{\rho}_z\frac{d\omega_0}{dt}\right)$$

$$\frac{d\omega_0}{dt}\left(\frac{d^2}{dz^2}-\lambda^2\right)^2W-\frac{1}{Pr}\frac{d^2\omega_0}{dt^2}\left(\frac{d^2}{dz^2}-\lambda^2\right)W=-\lambda^2\tau\left(\frac{\theta_0}{G_{cr}}\frac{dG}{dt}+\frac{G}{G_{cr}}\frac{d\theta_0}{dt}\right)$$

$$\frac{d^2\omega_0}{dt^2}\left(\frac{d^2}{dz^2}-\lambda^2\right)^2W-\frac{1}{Pr}\frac{d^3\omega_0}{dt^3}\left(\frac{d^2}{dz^2}-\lambda^2\right)W=-\lambda^2\tau\left(\frac{\theta_0}{G_{cr}}\frac{d^2G}{dt^2}+\frac{2}{G_{cr}}\frac{d\theta_0}{dt}\frac{dG}{dt}+\frac{G}{G_{cr}}\frac{d^2\theta_0}{dt}\right)$$

At $t=t_{cr}$, these equations reduce to

(a) $\theta_0(\tau_{zz}-\lambda^2\tau)=Ra_{cr}\omega_0\bar{\rho}_zW$

(b) $\omega_0\left(\frac{d^2}{dz^2}-\lambda^2\right)^2W=-\lambda^2\theta_0\tau$

(c) $-\frac{d^2\theta_0}{dt^2}\tau=Ra_{cr}W\omega_0\bar{\rho}_{zt}$

(d) $-\frac{1}{Pr}\frac{d^2\omega_0}{dt^2}\left(\frac{d^2}{dz^2}-\lambda^2\right)W=-\lambda^2\tau\frac{\theta_0}{G_{cr}}\frac{dG}{dt}$

(e) $\frac{d^2\omega_0}{dt^2}\left(\frac{d^2}{dz^2}-\lambda^2\right)^2W-\frac{1}{Pr}\frac{d^3\omega_0}{dt^3}\left(\frac{d^2}{dz^2}-\lambda^2\right)W=-\lambda^2\tau\left(\frac{\theta_0}{G_{cr}}\frac{d^2G}{dt^2}+\frac{d^2\theta_0}{dt^2}\right)$

Then, from (c) and (b),

$$\frac{1}{\theta_0}\frac{d^2\theta_0}{dt^2}W\left(\frac{d^2}{dz^2}-\lambda^2\right)^2W=Ra_{cr}\lambda^2W^2\bar{\rho}_{zt}$$

Since

$$J_1\equiv\int W\left(\frac{d^2}{dz^2}-\lambda^2\right)^2W\,dz=\int\left[\left(\frac{d^2W}{dz^2}\right)^2+2\lambda^2\left(\frac{dW}{dz}\right)^2+\lambda^4W^2\right]dz>0$$

then

$$\frac{1}{\theta_0}\frac{d^2\theta_0}{dt^2}=\frac{Ra_{cr}\lambda^2}{J_1}\int W^2\bar{\rho}_{zt}\,dz$$

Thus, the temperature perturbation passes into an unstable regime if $\bar{\rho}_{zt}>0$. Since the rate of heat transfer upwards is proportional to $\bar{\rho}_z$, the average weighted heat-transfer rate should be increasing. The most important region for this to be occurring is well in the body of the fluid, since the values near the boundaries are multiplied by W^2, which is very small in the regions adjacent to the walls. The Prandtl number Pr and the rate of change of the body force field dG/dt do not affect the rate of temperature transition to instability at the critical time $t=t_{cr}$.

For the slow-motion velocity, equation (d) is multiplied by W and integrated, and after elimination of τ by equation (b) there is obtained

$$\frac{1}{\omega_0}\frac{d^2\omega_0}{dt^2}=\frac{Pr}{G_{cr}}\frac{dG}{dt}\frac{J_1}{J_2}$$

where

$$J_2\equiv\int\left[\left(\frac{dW}{dz}\right)^2+\lambda^2W^2\right]dz>0$$

The rate of transition of the slow motion is, in constrast to the temperature distribution, proportional to the product of the Prandtl number and the rate of increase of the body force field and is independent of the rate of change of the density distribution. In the case where G is constant, the slow motion is stationary and continues in this condition while the temperature perturbation passes into instability. The behavior of the slow motion is in this case shown by means of a higher-order differential where equations (e) and (b) are combined:

$$\frac{1}{\omega_0}\frac{d^3\omega_0}{dt^3}=Pr\frac{J_1}{J_2}\frac{1}{\theta_0}\frac{d^2\theta_0}{dt^2}$$

The slow motion therefore passes into instability when G is constant, under the same conditions for the density distribution $\bar{\rho}(z,t_{cr})$ as when G is increasing. In this case the velocity persists temporarily at a constant amplitude, and the rate of transition is proportional to the product of the Prandtl number and the rate of transition of the temperature perturbation.

CALCULATION OF CRITICAL RAYLEIGH NUMBER

The differential equations are treated by a method analogous to that described by Chandrasekhar (ref. 12). First, the highest-order spatial derivatives of equations (25) and (26) are expanded in a sine series; these may be written

$$\left(\frac{\partial^2}{\partial z^2}-\lambda^2\right)^2\omega=-\lambda^2\sum\omega_n(t)\sin n\pi z$$

$$\left(\frac{\partial^2}{\partial z^2}-\lambda^2\right)\theta=-\sum(n^2\pi^2+\lambda^2)\theta_n(t)\sin n\pi z$$

When these two expansions are integrated term by term and the boundary conditions inserted for each term, there is obtained

$$\theta=\sum\theta_n(t)\sin n\pi z \tag{31}$$

and

$$\omega=\sum\omega_n(t)W_n(z) \tag{32}$$

where

$$\begin{aligned}W_n(z)=&-\frac{\lambda^2}{(n^2\pi^2+\lambda^2)^2}\sin n\pi z\\&+\left[A_n\sinh\lambda\left(z-\frac{1}{2}\right)+D_n\left(z-\frac{1}{2}\right)\cosh\lambda\left(z-\frac{1}{2}\right)\right]\\&+\left[B_n\left(z-\frac{1}{2}\right)\sinh\lambda\left(z-\frac{1}{2}\right)+C_n\cosh\lambda\left(z-\frac{1}{2}\right)\right]\end{aligned} \tag{33}$$

and

$$\begin{aligned}-\frac{A_n}{\cosh\lambda/2}=\frac{D_n}{2\sinh\lambda/2}\\&=\left[\frac{\lambda^2}{2(\sinh\lambda-\lambda)}\right]\left\{\frac{n\pi[(-)^n+1]}{(n^2\pi^2+\lambda^2)^2}\right\}\end{aligned}$$

$$\begin{aligned}\frac{B_n}{2\cosh\lambda/2}=-\frac{C_n}{\sinh\lambda/2}\\&=\left[\frac{\lambda^2}{2(\sinh\lambda+\lambda)}\right]\left\{\frac{n\pi[(-)^n-1]}{(n^2\pi^2+\lambda^2)^2}\right\}\end{aligned}$$

Substitution of these series into the differential equations yields

$$\left.\begin{aligned}\sum\frac{d\omega_n}{dt}\left(\frac{d^2}{dz^2}-\lambda^2\right)W_n\\=Pr\lambda^2\sum\left(\theta_n\frac{G}{G_{cr}}-\omega_n\right)\sin n\pi z\\\sum\left[(n^2\pi^2+\lambda^2)\theta_n+\frac{d\theta_n}{dt}\right]\sin n\pi z=Ra_{cr}\bar{\rho}_z\sum\omega_nW_n\end{aligned}\right\} \tag{34}$$

These equations are then multiplied by $\sin m\pi z$ and integrated; and the relation

$$\begin{aligned}K_{mn}\equiv&\int\sin m\pi zW_n(z)\,dz\\&=-\frac{1}{(m^2\pi^2+\lambda^2)}\int\sin m\pi z\left(\frac{d^2}{dz^2}-\lambda^2\right)W_n(z)\,dz\end{aligned}$$

is utilized to yield

$$\left.\begin{aligned}(m^2\pi^2+\lambda^2)\sum_n K_{mn}\frac{d\omega_n}{dt}\\=-\frac{1}{2}Pr\lambda^2\sum_n\left(\theta_n\frac{G}{G_{cr}}-\omega_n\right)\delta_m^n\quad m=1,2,3,\ldots\\\sum_n\left[\frac{d\theta_n}{dt}+(n^2\pi^2+\lambda^2)\theta_n\right]\delta_m^n=-2Ra_{cr}\sum_n\omega_nE_{mn}\end{aligned}\right\} \tag{35}$$

where

$$E_{mn}(t)\equiv\int\bar{\rho}_z\sin m\pi zW_n(z)\,dz$$

$$\delta_m^n=\begin{cases}0, & m\neq n\\1, & m=n\end{cases}$$

In the steady heating case ($\bar{\rho}_z=1$), $E_{mn}=K_{mn}$. At the critical time, the first of equations (35) reduces to

$$\theta_m(t_{cr})=\omega_m(t_{cr}) \tag{36}$$

and the second to

$$\sum_n[(n^2\pi^2+\lambda^2)\delta_m^n+2Ra_{cr}E_{mn}]\omega_n=0,$$
$$t=t_{cr},\ m=1,2,3,\ \ldots \tag{37}$$

For consistency, the determinant of the coefficients must be zero:

$$\det|(n^2\pi^2+\lambda^2)\delta_m^n+2Ra_{cr}E_{mn}|=0 \tag{38}$$

This equation determines the relation between Ra_{cr} and the shape of the basic density profile $\bar{\rho}$ at the critical time. The rate of change of temperature has no effect on the critical Rayleigh number other than to affect the shape of the density curve $\bar{\rho}$ at the critical time. Although the time scale could have been chosen in a different manner, it is noteworthy that the present choice, which is based on the natural heating period of the fluid layer and does not involve fluid viscosity, gives a Rayleigh number that is independent of the viscosity. Thus, the critical Rayleigh number may be said to be independent of the Prandtl number of the fluid. Also, the rate of variation of the body force G is not involved, but only its instantaneous value G_{cr} at the critical time t_{cr}.

Equation (38) will have a number of roots Ra_{cr} each one of which will determine with (37) an infinite sequence $\omega_n(t_{cr})/\omega_1(t_{cr})$. With the additional conditions (36) and

$$\frac{d\omega_n}{dt}=\frac{d\theta_n}{dt}=0,\ t=t_{cr}$$

this is sufficient to uniquely determine by means of (35) the sequence of functions $\omega_n(t)/\omega_1(t_{cr})$, $\theta_n(t)/\omega_1(t_{cr})$. The sums (31) and (32) then give the eigenfunctions ω and θ for each eigenvalue Ra_{cr}.

NUMERICAL RESULTS

To obtain numerical results, a definite density schedule at the lower wall ($\bar{\rho}(0,t)$) is required, such that $\bar{\rho}(0,0)=1$, $\bar{\rho}(0,t_{cr})=0$. A simple function of this type is

$$\bar{\rho}(0,t)=\frac{e^{-\beta t}-e^{-\beta t_{cr}}}{1-e^{-\beta t_{cr}}} \tag{39}$$

Then

$$\frac{d}{dt}\bar{\rho}(0,0)=\frac{-\beta}{1-e^{-\beta t_{cr}}}$$

$$\frac{d}{dt}\bar{\rho}(0,t_{cr})=\frac{-\beta e^{-\beta t_{cr}}}{1-e^{-\beta t_{cr}}}=e^{-\beta t_{cr}}\frac{d}{dt}\bar{\rho}(0,0)$$

For any fixed value of β, a decrease of t_{cr} to small values yields large values of the rate of density decrease at the lower wall, and there are developed density distributions $\bar{\rho}(z,t_{cr})$ with large gradients at the lower wall and small values in the remainder of the fluid. If β is not large, then when t_{cr} is small, βt_{cr} is small, and

$$\frac{d}{dt}\bar{\rho}(0,0)\approx-\frac{1}{t_{cr}}\approx\frac{d}{dt}\bar{\rho}(0,t_{cr})$$

Thus, when t_{cr} is small, small and moderate values of β have no significant influence. When βt_{cr} is not small, the effect of an increase in β is to increase the initial rate of density decrease at the lower wall, at the expense of the rate at the critical time. Thus, large values of β yield density distributions $\bar{\rho}(z,t_{cr})$ with more nearly uniform gradients than those for small values of β. Clearly, the limits $\beta\to 0$, $\beta\to\infty$ correspond respectively to uniform temperature rise at the lower wall and to sudden heating of the lower wall, with the latter type tending to more nearly linear density distributions.

Solution of the heat equation with the assumed boundary and initial conditions yields:

$$(1-e^{-\beta t_{cr}})\bar{\rho}_z(z,t)=1-e^{-\beta t}$$
$$+2\beta\sum_1^\infty\frac{\cos n\pi z}{n^2\pi^2-\beta}\ (e^{-\beta t}-e^{-n^2\pi^2 t}) \tag{40}$$

The functions E_{mn}, where

$$(1-e^{-\beta t_{cr}})E_{mn}=\int_0^1(1-e^{-\beta t_{cr}})\bar{\rho}_z\sin m\pi z W_n\,dz \tag{41}$$

are listed in table I with W_n for $m=1,\ 2,\ n=1,\ 2$. Solution of the determinant (38) yields values of Ra_{cr}, which are plotted in figure 2. Each point on these curves is the minimum value of Ra_{cr} from a curve of Ra_{cr} as a function of λ. Thus these are critical Rayleigh numbers for the most unstable cell shape. The cell-shape parameter for minimum

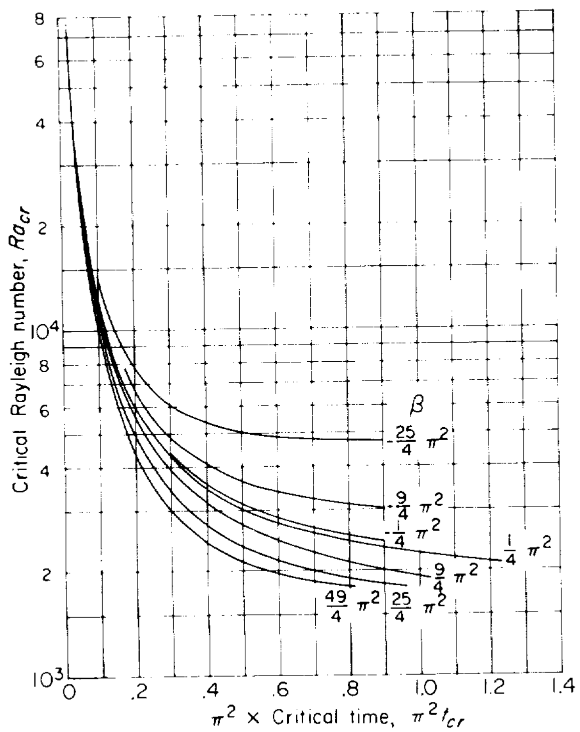


FIGURE 2.—Critical Rayleigh number dependence on critical time and heating rate.

value of Ra_{cr} is plotted in figure 3, and values are tabulated in table II. The trends of Ra_{cr} with t_{cr} and β both indicate that the effect of a large density gradient near the lower wall and a small gradient in the body of the fluid is to stabilize the fluid. Thus, if the lower wall temperature is being rapidly increased, cellular motion is delayed until Rayleigh numbers, hundreds of times the value for very slow heating, are reached. The trend with slow temperature change rates (t_{cr} large and $\beta>0$) is toward a value of $Ra_{cr}=1715$ (the value that is calculated with the second-order determinant—the true value is 1707.8). Variation in cell shape λ is seen to be slight.

Figure 2 can be used in a practical case where it is possible to fit well the density schedule $\bar{\rho}^*(0,t^*)$ at the lower wall by a curve of the type

$$\frac{\bar{\rho}^*(0,0)-\bar{\rho}^*(0,t)}{\bar{\rho}^*(0,0)}\equiv\frac{(1-e^{-\beta t})\Delta}{1-e^{-\beta t_{cr}}}$$

The value of β may be determined by the change of the slope $d\bar{\rho}^*/dt$ at time $t=t_1$ from $t=0$ as

$$\left(\frac{d\bar{\rho}^*}{dt}\right)_{t_1}=e^{-\beta t_1}\left(\frac{d\bar{\rho}^*}{dt}\right)_0$$

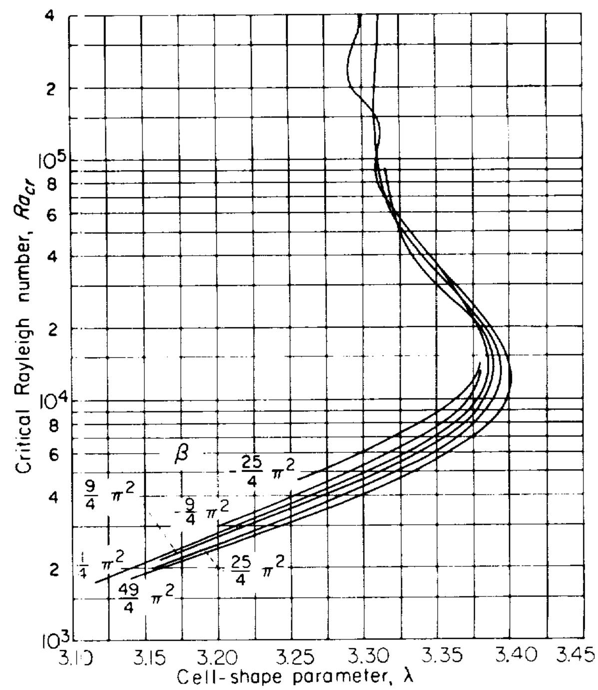


FIGURE 3.—Variation of cell-shape parameter with critical Rayleigh number.

The instantaneous value of the Rayleigh number is known at any time [$(\bar{\rho}^*(0,0)/\bar{\rho}^*(0,t)-1)G$ is substituted for $G_{cr}\Delta$], so that

$$\frac{Ra/G}{1-e^{-\beta t}}=\frac{Ra_1/G_1}{1-e^{-\beta t_1}}=\frac{Ra_{cr}/G_{cr}}{1-e^{-\beta t_{cr}}}$$

may be calculated. Thus, the intersection of the curve

$$Ra=\left(\frac{Ra_1}{1-e^{-\beta t_1}}\right)(1-e^{-\beta t})\frac{G}{G_1}$$

with the curve on figure 2 for the proper value of β identifies Ra_{cr} and t_{cr}.

SUMMARY OF RESULTS

For a layer of incompressible fluid in a varying body force field heated from below so that the density at the lower boundary varies with time by a relatively small amount, the following results were found for instability with respect to cellular motion:

1. The velocity resulting from the vertical expansion of the fluid does not affect its stability.
2. The critical Rayleigh number at which the fluid passes through neutral stability is determined

by the shape of the density curve at the instant of transition and is independent of past history, rate of change of the density profile, the Prandtl number, and rate of change of body force field.

3. For a nontrivial class of density distributions, the stationary condition on the slow-motion and temperature perturbations yields a neutral stability condition for a nonoscillatory motion that passes from a stable to an unstable condition at minimum Rayleigh number.

4. For rapid heating, the lower boundary will not benefit from internal, convective heat transfer until the temperature has increased well beyond that for attainment of internal circulation for very slow heating. The density differential was calculated as high as 600 times the static value.

5. The shape of the circulation cells for minimum critical Rayleigh number varies only slightly with heating rate.

6. The rate at which the density perturbation increases after the critical time is independent of the Prandtl number and of the rate of change of the gravitational field. The velocity increases at a rate proportional to the product of the Prandtl number and the rate of growth of the gravitational field. For a constant body force field, the perturbation growth rate is zero at the critical time but grows as a result of a higher-order time derivative that is proportional to the product of the Prandtl number and the rate of growth of the density perturbation.

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TABLE I.—LIST OF FUNCTIONS REQUIRED FOR STABILITY CONDITION

$$
\begin{aligned}
W_{2m-1}=\frac{\lambda^2}{[(2m-1)^2\pi^2+\lambda^2]^2}\Bigg\{\frac{(2m-1)\pi}{\sinh\lambda+\lambda}\left[\sinh\left(\frac{\lambda}{2}\right)\cosh\lambda\left(z-\frac{1}{2}\right)\right.\\
\left.-2\left(\cosh\frac{\lambda}{2}\right)\left(z-\frac{1}{2}\right)\sinh\lambda\left(z-\frac{1}{2}\right)\right]-\sin\,(2m-1)\pi z\Bigg\}
\end{aligned}
$$

$$
W_{2m}=\frac{\lambda^2}{(4\pi^2m^2+\lambda^2)^2}\left\{\frac{2m\pi}{\sinh\lambda-\lambda}\left[-\cosh\frac{\lambda}{2}\sinh\lambda\left(z-\frac{1}{2}\right)+2\left(\sinh\frac{\lambda}{2}\right)\left(z-\frac{1}{2}\right)\cosh\lambda\left(z-\frac{1}{2}\right)\right]-\sin 2m\pi z\right\}
$$

$$
(1-e^{-\beta t_{cr}})E_{11}=\frac{\lambda^2}{(\pi^2+\lambda^2)^2}\left[\begin{aligned}&\left[\frac{8\pi^2\lambda\cosh^2\frac{\lambda}{2}}{(\sinh\lambda+\lambda)(\pi^2+\lambda^2)^2}-\frac{1}{2}\right](1-e^{-\beta t})+\left(\frac{\beta}{2}\,\frac{e^{-\beta t}-e^{-4\pi^2t}}{4\pi^2-\beta}\right)\\&+\beta\left(\frac{8\pi^2\lambda\cosh^2\frac{\lambda}{2}}{\sinh\lambda+\lambda}\right)\left(\sum_{m=1}^{\infty}\left(\frac{e^{-\beta t}-e^{-4m^2\pi^2t}}{4m^2\pi^2-\beta}\right)\left\{\frac{2m+1}{[\lambda^2+(2m+1)^2\pi^2]^2}-\frac{2m-1}{[\lambda^2+(2m-1)^2\pi^2]^2}\right\}\right)\end{aligned}\right]
$$

$$
(1-e^{-\beta t_{cr}})E_{22}=\frac{\lambda^2}{(4\pi^2+\lambda^2)^2}\left\{\begin{aligned}&\left[-\frac{1}{2}+\frac{32\pi^2\lambda\sinh^2\frac{\lambda}{2}}{(\lambda^2+4\pi^2)^2(\sinh\lambda-\lambda)}\right](1-e^{-\beta t})+\left(\frac{\beta}{2}\,\frac{e^{-\beta t}-e^{-16\pi^2t}}{16\pi^2-\beta}\right)\\&-\beta\left(\frac{32\lambda\pi^2\sinh^2\frac{\lambda}{2}}{\sinh\lambda-\lambda}\right)\sum_{m=1}^{\infty}\left(\frac{e^{-\beta t}-e^{-4m^2\pi^2t}}{4m^2\pi^2-\beta}\right)\left\{\frac{m+1}{[\lambda^2+4(m+1)^2\pi^2]^2}-\frac{m-1}{[\lambda^2+4(m-1)^2\pi^2]^2}\right\}\end{aligned}\right\}
$$

$$
(1-e^{-\beta t_{cr}})E_{12}=\frac{2\beta\lambda^2}{(4\pi^2+\lambda^2)^2}\left\{\begin{aligned}&-\frac{1}{4}\,\frac{e^{-\beta t}-e^{-\pi^2t}}{\pi^2-\beta}+\frac{1}{4}\,\frac{e^{-\beta t}-e^{-9\pi^2t}}{9\pi^2-\beta}\\&+\frac{16\pi^2\lambda\sinh^2\frac{\lambda}{2}}{\sinh\lambda-\lambda}\sum_{m=1}^{\infty}\left[\frac{e^{-\beta t}-e^{-(2m-1)^2\pi^2t}}{(2m-1)^2\pi^2-\beta}\right]\left\{\frac{m}{(\lambda^2+4m^2\pi^2)^2}-\frac{m-1}{[\lambda^2+4(m-1)^2\pi^2]^2}\right\}\end{aligned}\right\}
$$

$$
(1-e^{-\beta t_{cr}})E_{21}=\frac{2\beta\lambda^2}{(\lambda^2+\pi^2)^2}\left\{\begin{aligned}&-\frac{1}{4}\,\frac{e^{-\beta t}-e^{-\pi^2t}}{\pi^2-\beta}+\frac{1}{4}\,\frac{e^{-\beta t}-e^{-9\pi^2t}}{9\pi^2-\beta}\\&+\frac{4\pi^2\lambda\cosh^2\frac{\lambda}{2}}{\sinh\lambda+\lambda}\sum_{m=1}^{\infty}\left[\frac{e^{-\beta t}-e^{-(2m-1)^2\pi^2t}}{(2m-1)^2\pi^2-\beta}\right]\left\{\frac{2m+1}{[\lambda^2+(2m+1)^2\pi^2]^2}-\frac{2m-3}{[\lambda^2+(2m-3)^2\pi^2]^2}\right\}\end{aligned}\right\}
$$

TABLE II.—PARAMETERS FOR NEUTRAL STABILITY

βt_{cr}	$\beta=\pi^2/4$		$\beta=9\pi^2/4$		$\beta=25\pi^2/4$		$\beta=49\pi^2/4$	
	$Ra_{cr}\times 10^{-3}$	λ	$Ra_{cr}\times 10^{-3}$	λ	$Ra_{cr}\times 10^{-3}$	λ	$Ra_{cr}\times 10^{-3}$	λ
$\beta>0$								
0.025	12.434	3.385	—	—	—	—	—	—
.05	6.0252	3.331	92.657	3.315	398.41	3.312	1061.9	3.215
.10	3.4929	3.241	34.751	3.345	146.14	3.308	383.04	3.299
.15	2.7609	3.196	19.995	3.380	81.370	3.312	211.86	3.291
.2	2.4262	3.172	13.816	3.390	53.707	3.324	139.11	3.313
.3	2.1243	3.150	8.5820	3.375	30.009	3.358	76.687	3.310
.4	1.9912	3.138	6.3389	3.345	20.052	3.382	50.121	3.332
.5	1.9185	3.131	5.1126	3.315	14.832	3.394	36.013	3.351
.75	1.8300	3.121	3.6172	3.255	8.8922	3.382	19.881	3.389
1.0	1.7893	3.118	—	—	6.3900	3.354	13.253	3.401
1.1	—	—	2.7570	3.205	—	—	—	—
1.25	1.7663	3.117	2.5509	3.190	5.0419	3.324	9.8258	3.398
1.50	1.7517	3.116	2.3110	3.175	4.2086	3.295	7.7879	3.383
1.75	1.7419	3.116	2.1505	3.160	3.6476	3.269	6.4573	3.364
2.0	—	—	2.0385	3.150	3.2474	—	5.5297	3.345
2.5	—	—	1.8990	3.140	2.7223	3.215	4.3364	3.310
3.0	—	—	—	—	2.4020	3.190	3.6146	3.280
3.5	—	—	—	—	2.1934	3.174	3.1401	3.250
4.0	—	—	—	—	2.0521	3.160	2.8103	3.230
6.0	—	—	—	—	1.8000	3.135	2.1539	3.175
8.0	—	—	—	—	—	—	1.9104	3.150
10.0	—	—	—	—	—	—	1.8046	3.140
$\beta<0$								
$-\beta t_{cr}$	$\beta=-\pi^2/4$		$\beta=-9\pi^2/4$		$\beta=-25\pi^2/4$			
0.1	12.559	3.385	13.080	3.38	14.245	3.38		
.3	4.3854	3.28	4.8382	3.29	6.0293	3.30		
.5	3.1459	3.22	3.6338	3.23	5.0410	3.27		
.7	2.6698	3.19	3.2008	3.21	4.7946	3.26		
.9	2.4270	3.17	2.9993	3.20	4.7255	3.255		

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National Aeronautics and Space Administration
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Arthur W. Goldstein. 1959. 15 p. diagrs., tabs.
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